<Contributed Talk 34>β-expansion's Attractors Observed in A/D converters

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β-expansion's Attractors Observed in A/D converters

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A new class of analog-to-digital (A/D) and digital-to-analog (D/A) converters using a flaky quantiser, called the β-encoder, as shown in Fig. 1 [1, 2, 3] has been shown to have exponential bit rate accuracy while possessing a self-correction property for fluctuations of the amplifier factor β and the quantiser threshold ν. Motivated by the close relationships [4, 5, 6] between β-transformations and β-expansion, we have recently observed[9, 10] that (1) such a flaky quantiser is exactly realized by the "*multi-valued Rényi-Parry map*", defined here so that probabilistic behavior in the "flaky region" is completely explained using dynamical systems theory; (2) a sample x is always confined to a subinterval of the contracted interval while the successive approximation of x is *stably* performed using β-expansion even if ν may vary at each iteration (*i.e.* a small real-valued quantity, approximately proportional to the quantisation error, does not necessarily converge to any fixed value, *e.g.,* 0 but may oscillate without diverging . Such a phenomenon is precisely the kind of "*chaos*"); (3) such a subinterval enables us to obtain the decoded sample easily, as it is equal to the midpoint of the subinterval and to prove two classic β-expansions, known as the *greedy* and *lazy* expansions [7, 8] are perfectly symmetrical in terms of their quantisation errors. The subinterval further suggests that ν should be set to around the midpoint of its associated greedy and lazy values. A switched-capacitor (SC) circuit technique [11, 12] has been proposed for implementing A/D convereter cicuit based on several types of β-encoders and SPICE simulations have been given to verify the validity of these circuits against deviations and mismatches of circuit parameters. Our review

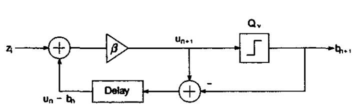

Figure 1: A typical β-encoder with its input $z_0 = y \in [0,1), z_i = 0, i > 1$ and output $(b_{i,\beta})_{i\geq 1}$.

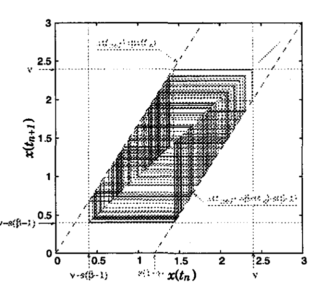

Figure 2: The scale-adjusted β-map: $S_{\beta,\nu,s}(x)$ and its eventually onto map in which an attractor can be observed.

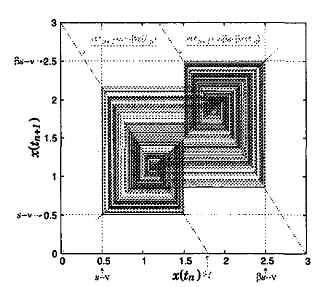

Figure 3: The scale-adjusted negative β-map: $R_{\beta,\nu,s}(x)$ and its eventually onto map in which an attractor can be observed.

is twofold. First, the β-encoder leads us to naturally define the "*multi-valued Rényi-Parry map*" [4, 5] with its eventually onto map, as it is identical to the Parry's $(\beta,\alpha)-$map [6]. Second, *chaos*, called "*β-expansion's attractors*" can be observed on the onto-map. Two types of β-expansion's attractors are as follows:

1. **Scale-Adjusted β-Map**[9, 11]: Daubechies et al. [1, 2] introduced a "flaky" version of an imperfect quantiser, defined as

$$Q^f_{\Delta_\beta}(z) = \begin{cases} 0, & \text{if } z \leq \nu_0, \\ 1, & \text{if } z \geq \nu_1, \\ 0 \text{ or } 1, & \text{if } z \in \Delta_\beta = [\nu_0, \nu_1], \nu_0 < \nu_1, \end{cases} \tag{1}$$

which is a ν-varying model of a quantiser $Q_\nu(z) = \begin{cases} 0, & \text{if } z \leq \nu, \\ 1, & \text{if } z \geq \nu, \end{cases}$ $\nu \in [\nu_0, \nu_1], \nu_0 < \nu_1$. We obtain:

Lemma 1[9]: *Let $S_{\beta,\nu,s}(x)$ be the scale-adjuted map with a scale s, defined by*

$$S_{\beta,\nu,s}(x) = \beta x - s(\beta-1)Q_{\gamma\nu}(x) = \begin{cases} \beta x, & x \in [0, \gamma\nu), \\ \beta x - s(\beta-1), & x \in [\gamma\nu, s), \end{cases} \quad \nu \in [s(\beta-1), s),\ s > 0 \tag{2}$$

which is refferred to as the "multi-valued Rényi-Parry map" on the flaky region $\Delta_\beta = [s(\beta-1), s]$ *and has its eventually onto Parry's* $(\beta,\alpha)-$*map [6] with the subinterval* $[\nu - s(\beta-1), \nu)$ *as shown in Fig.2. This map realises the flaky quantiser* $Q^f_{[s(1-\gamma),s\gamma]}(\cdot)$. *Let* $b_{i,S^i_{\beta,\nu_1^i,s}}$ *be its associated bit sequence for the threshold sequence* $\nu_1^L = \nu_1\nu_2\cdots\nu_L$*, defined by*

$$b_{i,S^i_{\beta,\nu_1^i,s}} = Q_{\gamma\nu}(S^{i-1}_{\beta,\nu_1^{i-1},s}(x)) = \begin{cases} 0, & S^{i-1}_{\beta,\nu_1^{i-1},s}(x) \in [0,\gamma\nu_i), \\ 1, & S^{i-1}_{\beta,\nu_1^{i-1},s}(x) \in [\gamma\nu_i, s). \end{cases} \quad (3)$$

Then we get $x = s(\beta-1)\sum_{i=1}^L b_{i,S^i_{\beta,\nu_1^i,s}}\gamma^i + \gamma^L S^L_{\beta,\nu_1^L,s}(x)$ *and its decoded value* $\widehat{x}_{L,S^L_{\beta,\nu_1^L,s}} = s(\beta-1)\sum_{i=1}^L b_{i,S^i_{\beta,\nu_1^i,s}}\gamma^i + \frac{s\gamma^L}{2}$.

2. **Negative β-Map**[10, 12]:We get

Lemma 2[10]: Let $R_{\beta,\nu,s}(x) : [0,s) \to [0,s)$, $s>0$ be the (scale-adjusted) negative β-map, defined by

$$R_{\beta,\nu,s}(x) = -\beta x + s[1+(\beta-1)Q_{\gamma\nu}(x)] = \begin{cases} s-\beta x, & x\in[0,\gamma\nu), \\ \beta s - \beta x, & x \in [\gamma\nu, s), \end{cases} \quad \nu \in [s(\beta-1), s] \quad (4)$$

which is another "*multi-valued Rényi-Parry map*" on the flaky region $\Delta_\beta = [s(\beta-1), s]$ realising $Q^f_{[s(1-\gamma),s\gamma]}(\cdot)$ and has its eventually onto Parry's $(\beta,\alpha)-$map [6] with the subinterval $[s-\nu, \beta s - \nu)$ as shown in Fig. 3. Let $b_{i,R^i_{\beta,\nu_1^i,s}}$ be the associated bit sequence for the threshold sequence ν_1^L, defined by

$$b_{i,R^i_{\beta,\nu_1^i,s}} = Q_{\gamma\nu}(R^{i-1}_{\beta,\nu_1^{i-1},s}(x)) = \begin{cases} 0, & R^{i-1}_{\beta,\nu_1^{i-1},s}(x) \in [0,\gamma\nu_i), \\ 1, & R^{i-1}_{\beta,\nu_1^{i-1},s}(x) \in [\gamma\nu_i, s). \end{cases} \quad (5)$$

Then we get $x = (-\gamma)^L R^L_{\beta,\nu_1^L,s}(x) - s\sum_{i=1}^L f_{i,R^i_{\beta,\nu_1^i,s}}(-\gamma)^i$ and its decoded value $\widehat{x}_{L,R^L_{\beta,\nu_1^L,s}} = s\{(-\gamma)^L/2 - \sum_{i=1}^L f_{i,R^i_{\beta,\nu_1^i,s}}(-\gamma)^i\}$. where $f_{i,R^i_{\beta,\nu_1^i,s}} = 1 + b_{i,R^i_{\beta,\nu_1^i,s}}(\beta-1)$. Such a negative β-expansion defines a new A/D converter called a *negative β-encoder* which facilitates the implementation of stable analog circuits. Figures 2 [11] and 3 [12] show a typical β-expansion's attractor of Eqs.(2) and (4), respectively.

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